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EFFECT OF THE ELECTRIC FIELD ON THE WAVE FLOW REGIMES OF A THIN FILM OF A VISCOUS DIELECTRIC FLUID

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Waves on the surface of a thin film of a viscous dielectric fluid flowing down the inner surface of one plate of a plane capacitor with alternating voltage applied is considered. It is shown that the volume forces acting from the inhomogeneous electric field are negligibly small in the case of long waves, and the influence of the electric field reduces to the influence of additional pressure onto the film surface. A model equation for determining the deviation of the film thickness from the undisturbed value is derived in the long-wave approximation. Some numerical solutions of this equation are given.

Key words: *down-flowing film, alternating electric field, dielectric, model equation, wave regimes.*

1. Mathematical Model. Let us consider a thin-film flow of a viscous dielectric fluid flowing down the inner surface of one plate of a plane capacitor aligned vertically in the gravity field. The flow is schematically shown in Fig. 1 together with the coordinate system used. Alternating voltage $V(t) = V_0 f(t)$ is applied to the capacitor plates [$f(t)$ is an arbitrary dimensionless smooth function of time whose maximum value is equal to unity]. In the long-wave approximation, the solution of this problem for small flow rates reduces to the solution of one equation for determining the deviation of the film thickness from its undisturbed value.

The system of the Navier–Stokes equations that describe this flow has the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \nu\Delta\boldsymbol{v} + \boldsymbol{g} + \frac{\boldsymbol{f}}{\rho},$$
$$\operatorname{div}\boldsymbol{v} = 0. \tag{1}$$

Here, $\boldsymbol{v}$ is the velocity, p is the pressure, ρ is the density, ν is the kinematic viscosity, t is the time, $\boldsymbol{g}$ is the acceleration due to gravity, $\boldsymbol{f}$ is the force acting on a unit volume of the dielectric fluid from the electric field [1]:

$$\boldsymbol{f} = \frac{\varepsilon_0}{2}\nabla\Big[E_1^2\rho\Big(\frac{\partial\varepsilon_1}{\partial\rho}\Big)_T\Big] - \frac{\varepsilon_0}{2}E_1^2\nabla\varepsilon_1,$$

ε_0 is the electric constant, ε_1 is the dielectric permeability of the fluid, and $\boldsymbol{E}_1$ is the intensity of the electric field in the fluid.

Assuming that the process is isothermal and the fluid is homogeneous, and using the notation $\chi_1 = \rho(\partial\varepsilon_1/\partial\rho)_T$, we obtain

$$\boldsymbol{f} = \varepsilon_0\chi_1\nabla E_1^2/2.$$

The surface of the plate where the film flows is subjected to the boundary condition

$$y = 0: \qquad \boldsymbol{v} = 0; \tag{2}$$

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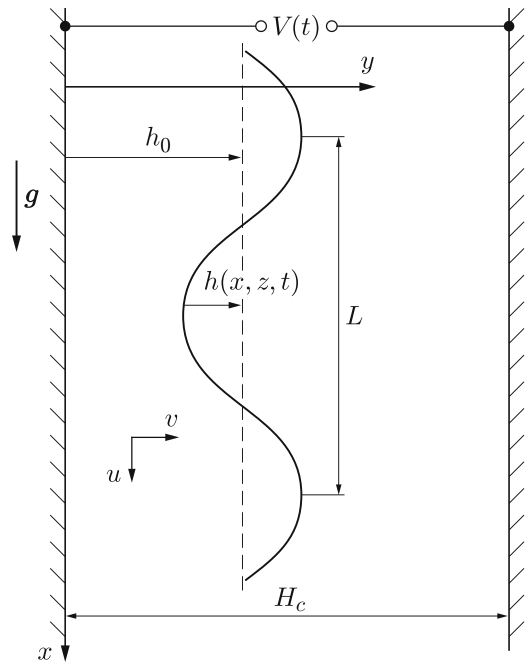

Fig. 1. Schematic of the flow.

the kinematic condition

$$\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + w\,\frac{\partial h}{\partial z} = v \tag{3}$$

and the dynamic conditions

$$y = h_0 + h: \qquad \sigma_{ik}^{(1)} n_k + (\sigma/R)\,\delta_{ik} n_k = \sigma_{ik}^{(2)} n_k \tag{4}$$

are imposed on the free surface of the film. In these equations, h_0 is the film thickness in the waveless flow, h is the deviation of the film thickness from the undisturbed value, u, v, and w are the x-, y-, and z-components of the velocity vector, n_i are the components of the normal to the film surface $\boldsymbol{n} = (n_x, n_y, n_z) = (-\partial h/\partial x, 1, -\partial h/\partial z)/\sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial z)^2}$, $1/R \approx -(\partial^2 h/\partial x^2 + \partial^2 h/\partial z^2)$ is the mean curvature (the film surface is assumed to be weakly curved, and the deviation h is assumed to be small), σ is the surface tension of the fluid, $\sigma_{ik} = -p\delta_{ik} + \sigma'_{ik} - \varepsilon_0(\varepsilon - \chi)E^2\delta_{ik}/2 + E_i D_k$ is the stress tensor, and σ'_{ik} is the tensor of viscous stresses, and $\boldsymbol{D} = \varepsilon_0\varepsilon\boldsymbol{E}$ is the electric field induction; the subscripts 1 and 2 refer to the fluid and the gas above the fluid, respectively.

System (1) with the boundary conditions (2)–(4) admits a solution with a constant-thickness plane free boundary

$$u(y) \equiv U = \frac{g}{2\nu}\,(2h_0 y - y^2) = U_s\Big(2\,\frac{y}{h_0} - \Big(\frac{y}{h}\Big)^2\Big),$$

$$P = p_2 - \frac{\varepsilon_0 V_0^2[\varepsilon_1^2(\varepsilon_2 + \chi_2) - \varepsilon_2^2(\varepsilon_1 + \chi_1)]}{2[(H_c - h_0)\varepsilon_1 + \varepsilon_2 h_0]^2}\, f^2(t).$$

Here, U_s is the velocity of the film surface, H_c is the distance between the capacitor plates, p_2 is the pressure of the gas above the film, and ε_2 is the dielectric permeability of the gas; the quantity χ_2 characterizing the gas is similar to the quantity χ_1.

We can easily demonstrate, however, that such a flow is unstable to infinitesimal disturbances and becomes wavy as these disturbances evolve.

The objective of the present work was to obtain a simplified model providing a qualitative description of nonlinear wave regimes in a down-flowing film of a dielectric fluid.

Assuming the deviation of the film thickness from its undisturbed value $h(x,z,t)$ to be small, we find a quasi-steady approximation of the potential distribution between the capacitor plates. Denoting the potential in the liquid film by $\varphi_1(x,y,z,t)$ and the potential in the gas by $\varphi_2(x,y,z,t)$, we obtain the following boundary-value problem:

$$0 \le y \le h_0 + h\colon \quad \Delta\varphi_1 = 0, \qquad h_0 + h \le y \le H_c\colon \quad \Delta\varphi_2 = 0. \tag{5}$$

The boundary conditions on the capacitor plates are

$$y = 0\colon \quad \varphi_1 = 0, \qquad y = H_c\colon \quad \varphi_2 = V(t), \tag{6}$$

and the film surface is subjected to the conditions

$$y = h_0 + h\colon \quad \boldsymbol{\tau}_1 \cdot \nabla\varphi_1 = \boldsymbol{\tau}_1 \cdot \nabla\varphi_2, \quad \boldsymbol{\tau}_2 \cdot \nabla\varphi_1 = \boldsymbol{\tau}_2 \cdot \nabla\varphi_2, \quad \varepsilon_1 \boldsymbol{n} \cdot \nabla\varphi_1 = \varepsilon_2 \boldsymbol{n} \cdot \nabla\varphi_2. \tag{7}$$

In Eqs. (5)–(7), we have $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$; $\boldsymbol{\tau}_1 = (1, \partial h/\partial x, 0)/\sqrt{1 + (\partial h/\partial x)^2}$ and $\boldsymbol{\tau}_2 = (0, \partial h/\partial z, 1)/\sqrt{1 + (\partial h/\partial z)^2}$ are the tangential vectors to the film surface, which lie in the planes xy and zy, respectively.

Using the parameters h_0 and λ for scaling in the y and x, z directions, respectively, using V_0 for scaling the potential, and assuming that the dimensionless quantity h is equal to $\varepsilon h'$ (in what follows, the prime is omitted), we write Eqs. (5) with the boundary conditions (6) and (7) in a dimensionless form with accuracy to terms of the order of $\varepsilon = h_0/\lambda$ inclusive:

$$\begin{gathered}
0 \le y \le 1 + \varepsilon h\colon \quad \frac{\partial^2\varphi_1}{\partial y^2} = 0; \qquad 1 + \varepsilon h \le y \le H_c\colon \quad \frac{\partial^2\varphi_2}{\partial y^2} = 0,\\
y = 0\colon \quad \varphi_1 = 0; \qquad y = H_c\colon \quad \varphi_2 = f(t);\\
y = 1 + \varepsilon h\colon \quad \frac{\partial\varphi_1}{\partial x} + \varepsilon\,\frac{\partial h}{\partial x}\,\frac{\partial\varphi_1}{\partial y} = \frac{\partial\varphi_2}{\partial x} + \varepsilon\,\frac{\partial h}{\partial x}\,\frac{\partial\varphi_2}{\partial y},\\
\frac{\partial\varphi_1}{\partial z} + \varepsilon\,\frac{\partial h}{\partial z}\,\frac{\partial\varphi_1}{\partial y} = \frac{\partial\varphi_2}{\partial z} + \varepsilon\,\frac{\partial h}{\partial z}\,\frac{\partial\varphi_2}{\partial y}, \qquad \varepsilon_1\,\frac{\partial\varphi_1}{\partial y} = \varepsilon_2\,\frac{\partial\varphi_2}{\partial y}.
\end{gathered} \tag{8}$$

Expanding φ_1 and φ_2 into series with respect to the small parameter ε and shifting the boundary conditions (8) to the undisturbed level $y = 1$, we obtain the following relations in the first approximation:

$$\varphi_1 = \frac{\varepsilon_2 f(t) y}{(H_c - 1)\varepsilon_1 + \varepsilon_2}\Big(1 + \varepsilon\,\frac{\varepsilon_1 - \varepsilon_2}{(H_c - 1)\varepsilon_1 + \varepsilon_2}\,h\Big),$$

$$\varphi_2 = \frac{\varepsilon_1 f(t)(y - H_c)}{(H_c - 1)\varepsilon_1 + \varepsilon_2}\Big(1 + \varepsilon\,\frac{\varepsilon_1 - \varepsilon_2}{(H_c - 1)\varepsilon_1 + \varepsilon_2}\,h\Big) + f(t).$$

Here, H_c is the dimensionless (in the units of h_0) distance between the capacitor plates.

Using the parameters λ/U_s, U_s, and $\rho g h_0$ as scales of time, velocity, and pressure, in addition to the parameters chosen above, and retaining terms of the order not higher than ε, we write the equations of motion of the film in a dimensionless form:

$$\begin{gathered}
\varepsilon\,\frac{\partial u}{\partial t} + \varepsilon u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + \varepsilon w\,\frac{\partial u}{\partial z} = -\frac{\varepsilon}{\mathrm{Fr}}\,\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Fr}} + \frac{1}{\mathrm{Re}}\,\frac{\partial^2 u}{\partial y^2} + W_{\mathrm{el}} A(t)\varepsilon^2\,\frac{\partial h}{\partial x},\\
\varepsilon\,\frac{\partial v}{\partial t} + \varepsilon u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + \varepsilon w\,\frac{\partial v}{\partial z} = -\frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\,\frac{\partial^2 v}{\partial y^2},\\
\varepsilon\,\frac{\partial w}{\partial t} + \varepsilon u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + \varepsilon w\,\frac{\partial w}{\partial z} = -\frac{\varepsilon}{\mathrm{Fr}}\,\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\,\frac{\partial^2 w}{\partial y^2} + W_{\mathrm{el}} A(t)\varepsilon^2\,\frac{\partial h}{\partial z},\\
\varepsilon\,\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \varepsilon\,\frac{\partial w}{\partial z} = 0.
\end{gathered} \tag{9}$$

Here, $\mathrm{Re} = U_s h_0/\nu$ is the Reynolds number, $\mathrm{Fr} = U_s^2/(gh_0)$ is the Froude number, $W_{\mathrm{el}} = \varepsilon_0\chi_1 V_0^2/(\rho U_s^2 h_0^2)$, and $A(t) = \varepsilon_2^2(\varepsilon_1-\varepsilon_2)f^2(t)/[(H_c-1)\varepsilon_1+\varepsilon_2]^3$. In the first and third equations of system (9), the last terms proportional to ε^2 are retained until the order of the product $W_{\mathrm{el}}A(t)$ is clarified.

Using V_0/h_0 as a scale of the electric field intensity and taking into account that $E_2^2 = E_{2n}^2 + E_{2\tau}^2$ and also $\varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$ and $E_{2\tau} = E_{1\tau}$ by virtue of the boundary conditions, we write the boundary condition (4) in projections onto the normal $\boldsymbol{n}$ in a dimensionless form (terms of the order of ε^2 and higher are neglected):

$$y = 1+\varepsilon h\colon\quad p = p_2 - \varepsilon\,\mathrm{We}\,\varepsilon^2\Delta h - B_{\mathrm{el}}\,\frac{\varepsilon_1^2(\varepsilon_2+\chi_2)-\varepsilon_2^2(\varepsilon_1+\chi_1)}{\varepsilon_2^2}\Big(\frac{\partial\varphi_1}{\partial y}\Big)^2 + 2\,\frac{\mathrm{Fr}}{\mathrm{Re}}\,\frac{\partial v}{\partial y}. \tag{10}$$

Here $\mathrm{We} = \sigma/(\rho g h_0^2)$ is the Weber number and $B_{\mathrm{el}} = \varepsilon_0 V_0^2/(2\rho g h_0^3)$.

As the equality $\varepsilon_0\varepsilon E_i E_k n_k \tau_i = E_i D_k n_k \tau_i = E_\tau D_n$ is valid for any vector $\boldsymbol{\tau}$ tangential to the film surface and $E_\tau^{(1)}D_n^{(1)} = E_\tau^{(2)}D_n^{(2)}$ for the electric field components by virtue of the boundary conditions, the boundary condition (4) in projections onto the directions $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ acquires the form

$$\sigma'_{ik}n_k\tau_{1i} = 0, \qquad \sigma'_{ik}n_k\tau_{2i} = 0. \tag{11}$$

After normalization, the boundary conditions (11) are written in the following form (terms of the order of ε^2 and higher are neglected):

$$y = 1+\varepsilon h\colon\qquad \frac{\partial u}{\partial y} + \varepsilon\,\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial w}{\partial y} + \varepsilon\,\frac{\partial v}{\partial z} = 0. \tag{12}$$

At $h = 0$, system (9) with no-slip conditions (2) and conditions (10), (12) on the free boundary describes a problem of an undisturbed flow whose solution has the form

$$v = w = 0, \qquad u \equiv U_0 = \frac{\mathrm{Re}}{\mathrm{Fr}}\Big(y - \frac{y^2}{2}\Big), \qquad p \equiv P_0 = p_2 - P_{\mathrm{el}}f^2(t).$$

Here,

$$P_{\mathrm{el}} = B_{\mathrm{el}}\,\frac{\varepsilon_1^2(\varepsilon_2+\chi_2)-\varepsilon_2^2(\varepsilon_1+\chi_1)}{[(H_c-1)\varepsilon_1+\varepsilon_2]^2}$$

is the maximum dimensionless value of the electric pressure jump. The kinematic condition on the free surface is written as

$$y = 1+\varepsilon h\colon\qquad \varepsilon^2\Big(\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + w\,\frac{\partial h}{\partial z}\Big) = v. \tag{13}$$

Assuming that $u = U_0 + \varepsilon u'$, $w = \varepsilon w'$, $v = \varepsilon^2 v'$, and $p = P_0 + \varepsilon p'$, we present system (9) with the boundary conditions (10), (12), and (13) in the form

$$\begin{gathered}\varepsilon\Big(\frac{\partial u'}{\partial t} + U_0\,\frac{\partial u'}{\partial x} + v'\,\frac{\partial U_0}{\partial y}\Big) = -\frac{\varepsilon}{\mathrm{Fr}}\,\frac{\partial p'}{\partial x} + \frac{1}{\mathrm{Re}}\,\frac{\partial^2 u'}{\partial y^2}, \qquad 0 = -\frac{1}{\mathrm{Fr}}\,\frac{\partial p'}{\partial y} + \frac{\varepsilon}{\mathrm{Re}}\,\frac{\partial^2 v'}{\partial y^2},\\ \varepsilon\Big(\frac{\partial w'}{\partial t} + U_0\,\frac{\partial w'}{\partial x}\Big) = -\frac{\varepsilon}{\mathrm{Fr}}\,\frac{\partial p'}{\partial z} + \frac{1}{\mathrm{Re}}\,\frac{\partial^2 w'}{\partial y^2}, \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0;\end{gathered} \tag{14}$$

$$\begin{gathered}y = 1+\varepsilon h\colon\qquad p' = -\,\mathrm{We}\,\varepsilon^2\,\Delta h - \frac{2(\varepsilon_1-\varepsilon_2)}{(H_c-1)\varepsilon_1+\varepsilon_2}\,P_{\mathrm{el}}f^2(t)h\\ -\Big(\frac{\varepsilon_1-\varepsilon_2}{(H_c-1)\varepsilon_1+\varepsilon_2}\Big)^2 P_{\mathrm{el}}f^2(t)h^2\varepsilon + 2\,\frac{\mathrm{Fr}}{\mathrm{Re}}\,\frac{\partial v'}{\partial y}\,\varepsilon,\\ \frac{\partial U_0}{\partial y} + \varepsilon\,\frac{\partial u'}{\partial y} = 0, \qquad \frac{\partial w'}{\partial y} = 0, \qquad \frac{\partial h}{\partial t} + (U_0+\varepsilon u')\,\frac{\partial h}{\partial x} + \varepsilon w'\,\frac{\partial h}{\partial z} = v'.\end{gathered} \tag{15}$$

To estimate the order of magnitude of terms proportional to P_{el}, we have to know the explicit expression for χ_1, which depends on the type of the fluid. If the dielectric fluid consists of non-polar molecules, we can use the Clausius–Mossotti approximation [2]

$$\frac{3(\varepsilon_1-1)}{\varepsilon_1+2} = \frac{\alpha N_{\mathrm{A}}}{m_0}\,\rho,$$

where α is the molecule susceptibility, N_{A} is the Avogadro number, and m_0 is the molecule weight. Then, we obtain

$$\chi_1 = \rho \frac{\partial \varepsilon_1}{\partial \rho} = \frac{(\varepsilon_1 - 1)(\varepsilon_1 + 2)}{3}.$$

We have $\varepsilon_1 \approx 2$ for non-polar fluids, whence it follows that $\chi_1 \approx 4/3$.

For polar fluids, we use the Onsager approximation, which implies that [2]

$$\frac{(\varepsilon_1 - n_1^2)(2\varepsilon_1 + n_1^2)}{\varepsilon_1 (n_1^2 + 2)^2} = \frac{\rho p^2}{9kTm_0\varepsilon_0}$$

(n_1 is the refractive index of the fluid, p is the own dipole moment of the molecule, k is the Boltzmann constant, and T is the absolute temperature). It follows from here that

$$\chi_1 = \rho \frac{\partial \varepsilon_1}{\partial \rho} = \varepsilon_1 \frac{(1 - n_1^2/\varepsilon_1)(2 + n_1^2/\varepsilon_1)}{2 + (n_1^2/\varepsilon_1)^2}.$$

As $n_1 \ll \varepsilon_1$ for polar fluids, then we have $\chi_1 \approx \varepsilon_1$ for them.

Concerning the gas above the film, we can assume that $\varepsilon_2 \approx 1$ and $\chi_2 \approx 0$ if the gas density is sufficiently low.

The estimates show that the volume electric force is negligibly small for thin films of both non-polar and polar fluids [for this reason, system (14) does not include terms proportional to $W_{\mathrm{el}}A(t)$]. This conclusion is also valid for terms proportional to P_{el} in the boundary condition (15) if the fluid consists of non-polar molecules. For polar fluids, these terms should be retained because they are of the same order as the Laplace pressure.

Based on the last comment, we assume that the fluid consists of polar molecules (water, alcohols, etc.). Shifting the boundary conditions (15) to the undisturbed level $y = 1$, we obtain

$$\begin{gathered}
y = 1: \qquad p' + \varepsilon \frac{\partial p'}{\partial y} h = -\mathrm{We}\,\varepsilon^2 \Delta h - \frac{2(\varepsilon_1 - \varepsilon_2)}{(H_c - 1)\varepsilon_1 + \varepsilon_2} P_{\mathrm{el}} f^2(t) h \\
- \Big(\frac{\varepsilon_1 - \varepsilon_2}{(H_c - 1)\varepsilon_1 + \varepsilon_2}\Big)^2 P_{\mathrm{el}} f^2(t) h^2 \varepsilon + 2\,\frac{\mathrm{Fr}}{\mathrm{Re}} \frac{\partial v'}{\partial y} \varepsilon, \\
\frac{\partial^2 U_0}{\partial y^2} h + \frac{\partial u'}{\partial y} + \varepsilon \frac{\partial^2 u'}{\partial y^2} h = 0, \qquad \frac{\partial w'}{\partial y} + \varepsilon \frac{\partial^2 w'}{\partial y^2} h = 0, \\
\frac{\partial h}{\partial t} + (U_0 + \varepsilon u') \frac{\partial h}{\partial x} + \varepsilon w' \frac{\partial h}{\partial z} = v' + \varepsilon \frac{\partial v'}{\partial y} h.
\end{gathered} \tag{16}$$

The solution of problem (14), (16) is sought in the form

$$(u', v', w', p') = \sum_{n=0}^{\infty} \varepsilon^n (u_n, v_n, w_n, p_n).$$

In the zeroth approximation in terms of ε, Eq. (14) yields

$$\frac{\partial^2 u_0}{\partial y^2} = 0, \qquad \frac{\partial p_0}{\partial y} = 0,$$

$$\frac{\partial^2 w_0}{\partial y^2} = 0, \qquad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial y} = 0.$$

From the boundary conditions (16), we obtain

$$\begin{gathered}
y = 1: \qquad p_0 = -\mathrm{We}\,\varepsilon^2 \Delta h - \frac{2(\varepsilon_1 - \varepsilon_2)}{(H_c - 1)\varepsilon_1 + \varepsilon_2} P_{\mathrm{el}} f^2(t) h, \\
\frac{\partial u_0}{\partial y} = -\frac{\partial^2 U_0}{\partial y^2} h, \qquad \frac{\partial w_0}{\partial y} = 0, \qquad \frac{\partial h}{\partial t} + U_0 \frac{\partial h}{\partial x} = v_0.
\end{gathered} \tag{17}$$

In this approximation, the solution of the problem has the form

$$p_0 = -\,\mathrm{We}\,\varepsilon^2\,\Delta h - \frac{2(\varepsilon_1-\varepsilon_2)}{(H_c-1)\varepsilon_1+\varepsilon_2}\,P_{\mathrm{el}} f^2(t)h, \quad u_0 = \frac{\mathrm{Re}}{\mathrm{Fr}}\,yh, \quad w_0 = 0, \quad v_0 = -\frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{y^2}{2}\,\frac{\partial h}{\partial x}.$$

Here, we take into account that $\partial^2 U_0/\partial y^2 = -\,\mathrm{Re}\,/\,\mathrm{Fr}$.

As $U_s = (1/2)\,\mathrm{Re}\,/\,\mathrm{Fr} = 1$, then we can substitute the resultant solution into the kinematic condition (17) and obtain

$$\frac{\partial h}{\partial t} + 2\,\frac{\partial h}{\partial x} = 0. \tag{18}$$

Thus, in the zeroth approximation in terms of ε, the electric field does not affect the evolution of film-thickness disturbances, and their velocity is equal to the doubled velocity of the film surface in a waveless flow. Introducing a "slow" time $\tau_1 = \varepsilon t$ and presenting the dependences of all quantities on time in the form $(u, v, w, p, h) = (u(x,z,t,\tau_1), v(x,z,t,\tau_1), w(x,z,t,\tau_1), p(x,z,t,\tau_1), h(x,z,t,\tau_1))$, we obtain the following system of equations of motion in the next approximation in terms of ε:

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{\partial p_0}{\partial x} + \mathrm{Re}\Big(\frac{\partial u_0}{\partial t} + U_0\,\frac{\partial u_0}{\partial x} + v_0\,\frac{\partial U_0}{\partial y}\Big),$$

$$\frac{\partial p_1}{\partial y} = \frac{\mathrm{Fr}}{\mathrm{Re}}\,\frac{\partial^2 v_0}{\partial y^2}, \qquad \frac{\partial^2 w_1}{\partial y^2} = \frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{\partial p_0}{\partial z}, \tag{19}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0.$$

In this approximation, the boundary conditions take the form

$$p_1 = -\Big(\frac{\varepsilon_1-\varepsilon_2}{(H_c-1)\varepsilon_1+\varepsilon_2}\Big)^2 P_{\mathrm{el}} f^2(t)h^2 + 2\,\frac{\mathrm{Fr}}{\mathrm{Re}}\,\frac{\partial v_0}{\partial y},$$

$$\frac{\partial u_1}{\partial y} = -\frac{\partial^2 u_0}{\partial y^2}\,h = 0, \qquad \frac{\partial w_1}{\partial y} = -\frac{\partial^2 w_0}{\partial y^2}\,h = 0, \tag{20}$$

$$\frac{\partial h}{\partial \tau_1} + u_0\,\frac{\partial h}{\partial x} = v_1 + \frac{\partial v_0}{\partial y}\,h.$$

System (19) with the boundary conditions (20) is easily integrated; the corresponding manipulations are not given here because they are too cumbersome. The expression for v_1 at $y = 1$ is

$$v_1(1) = \frac{3}{10}\,\mathrm{Re}\,\frac{\partial^2 h}{\partial x^2} + \frac{5}{12}\,\mathrm{Re}\,\frac{\partial^2 h}{\partial t\,\partial x} - \frac{4}{3}\,\frac{\varepsilon_1-\varepsilon_2}{(H_c-1)\varepsilon_1+\varepsilon_2}\,P_{\mathrm{el}} f^2(t)\,\Delta h - \frac{2}{3}\,\mathrm{We}\,\varepsilon^2\,\Delta^2 h. \tag{21}$$

Substituting Eq. (21) into the kinematic condition (20), we obtain

$$\frac{\partial h}{\partial \tau_1} + 4h\,\frac{\partial h}{\partial x} - \frac{3}{10}\,\mathrm{Re}\,\frac{\partial^2 h}{\partial x^2} - \frac{5}{12}\,\mathrm{Re}\,\frac{\partial^2 h}{\partial t\,\partial x}$$

$$+ \frac{4}{3}\,\frac{\varepsilon_1-\varepsilon_2}{(H_c-1)\varepsilon_1+\varepsilon_2}\,P_{\mathrm{el}} f^2(t)\,\Delta h + \frac{2}{3}\,\mathrm{We}\,\varepsilon^2\,\Delta^2 h = 0, \tag{22}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad \Delta^2 = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\Big)^2.$$

With allowance for Eq. (18), Eq. (22) is written in the form

$$\frac{\partial h}{\partial \tau_1} + 4h\,\frac{\partial h}{\partial x} + \frac{8}{15}\,\mathrm{Re}\,\frac{\partial^2 h}{\partial x^2} + \frac{4}{3}\,\frac{\varepsilon_1-\varepsilon_2}{(H_c-1)\varepsilon_1+\varepsilon_2}\,P_{\mathrm{el}} F(\tau_1)\,\Delta h + \frac{2}{3}\,\mathrm{We}\,\varepsilon^2\,\Delta^2 h = 0, \tag{23}$$

where $F(\tau_1) = f^2(\tau_1/\varepsilon)$.

Thus, Eq. (23) describes the evolution of spatial disturbances on a dielectric fluid film flowing down a vertical plane. Let us specify the chosen characteristic scale λ. We require identical absolute values of the coefficients at the third and fifth terms in Eq. (23). Then, for the small parameter ε used in the expansion, we have

$$\varepsilon = \sqrt{4\,\mathrm{Re}\,/(5\,\mathrm{We})}\,,$$

correspondingly, the characteristic longitudinal size of disturbances is determined by the equality

$$\lambda = \sqrt{5\,\mathrm{We}\,/(4\,\mathrm{Re})}\;h_0.$$

It follows from these relations that the assumption about the long-wave character of the disturbances considered is valid for high values of the Weber number.

As a result, using the substitution

$$h = aH, \qquad \tau = a\tau_1, \qquad a = (8/15)\,\mathrm{Re},$$

we transform Eq. (23) to

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + C\Phi(\tau)\,\Delta H + \Delta^2 H = 0, \tag{24}$$

where

$$C = \frac{5}{2\,\mathrm{Re}}\,\frac{\varepsilon_1 - \varepsilon_2}{(H_c - 1)\varepsilon_1 + \varepsilon_2}\,P_{\mathrm{el}}.$$

It follows from Eq. (18) that Eq. (24) is written in a reference system moving with respect to the wall and having a velocity equal to the doubled velocity of the film flow on the undisturbed free surface. We consider only two-dimensional solutions $H = H(x,\tau)$ in this paper. Then, Eq. (24) acquires the form

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + (1 + C\Phi(\tau))\,\frac{\partial^2 H}{\partial x^2} + \frac{\partial^4 H}{\partial x^4} = 0. \tag{25}$$

If the function $\Phi(\tau)$ is a sinusoidal function, Eq. (25) coincides with the equation obtained in [3, 4] in considering other types of film flows. If the difference of the potentials between the capacitor plates is equal to zero, then we have $C = 0$ and Eq. (25) transforms to an equation that describes the evolution of two-dimensional disturbances in a usual viscous fluid film flowing down a vertical wall. This equation is currently known as the Kuramoto–Sivashinsky equation, though it was first derived in [5] as applied to film flows. This equation is studied in many publications, which shows that it has a large variety of solutions.

2. Method of the Solution. If the nonlinear term in Eq. (25) is neglected, then it describes the stability of the initial flow to infinitesimal disturbances. It should be noted that the neutral wavenumber for the Kuramoto–Sivashinsky equation ($C = 0$) is $\alpha_n = 1$, and unstable disturbances are those with wavenumbers $\alpha < \alpha_n = 1$.

To analyze nonlinear periodic solutions, we solved Eq. (25) numerically. The solution was presented as a spatial Fourier series whose harmonics are functions of time:

$$H(x,\tau) = \sum_{n=-\infty}^{\infty} H_n(\tau)\exp{(in\alpha x)}. \tag{26}$$

As H is real, the relation $H_{-n} = H_n^*$ is valid for harmonics H_n (the asterisk indicates the complex conjugation operation).

Substituting Eq. (26) into Eq. (25), we obtain an infinite system of nonlinear ordinary differential equations for harmonics H_n. Terminating series (26), i.e., assuming that all harmonics beginning from a certain number N are equal to zero, we obtain a finite-dimensional approximation of this system:

$$\frac{dH_n}{d\tau} = \alpha^2 n^2 (1 + C\Phi(\tau) - \alpha^2 n^2) H_n - 4i\alpha n \sum_{m=N-n}^{N} H_m H_{n-m}, \qquad n = 0, 1, \ldots, N. \tag{27}$$

It follows from Eq. (27) that the equation for the zeroth harmonic is trivial, i.e., it is valid for an arbitrary constant value of H_0. In what follows, we construct solutions where this constant is equal to zero.

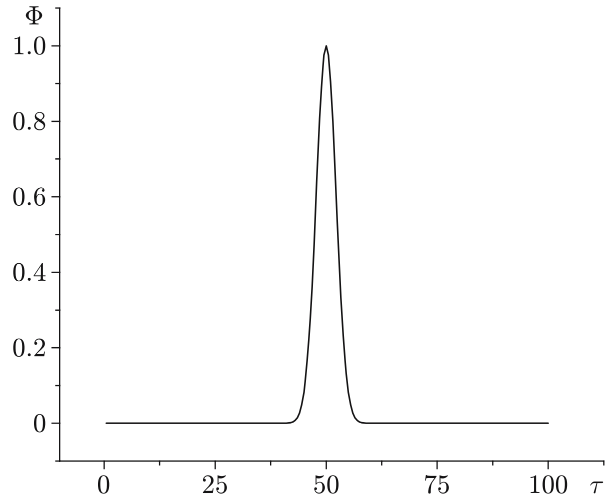

Fig. 2

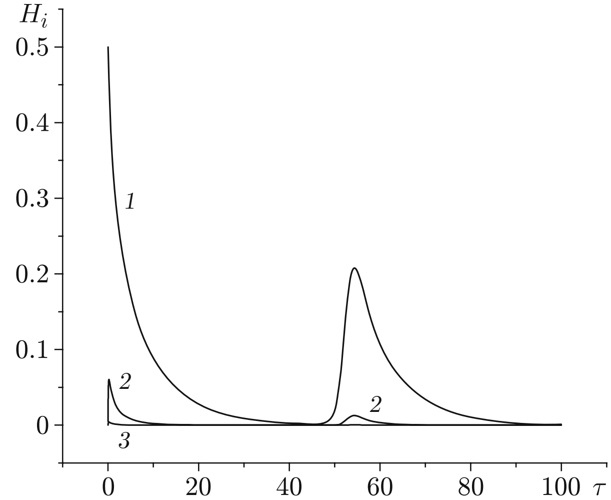

Fig. 3

Fig. 2. Form of the function $\Phi(\tau)$ in Eq. (25) at $C = 1$ and $\beta = 0.1$.

Fig. 3. Amplitudes of the first three harmonics versus time at $\alpha = 1.05$ and H_1 (1), H_2 (2), and H_3 (3).

Separating the real and imaginary parts in Eq. (27), we obtain a system of $2N$ ordinary differential equations for $2N$ unknown functions.

System (27) was solved numerically by one of the fifth-order Runge–Kutta methods with an automatic choice of the integration step and a controlled error. The number of harmonics was chosen to satisfy the relation $|H_N|/\max|H_n| < 10^{-4}$.

3. Calculation Results. As was noted above, Eq. (25) at $C = 0$ transforms to the known Kuramoto–Sivashinsky equation. The set of its solutions is extremely diversified. For instance, it was shown [5] that a family of periodic solutions of the Kuramoto–Sivashinsky equation existing at $\alpha \leq 0.4979$ is branched off from the trivial solution $H = 0$ if the wavenumber is $\alpha = \alpha_n = 1$ (neutral wavenumber). A cascade of bifurcations, which gave rise to more and more complicated solutions, was obtained in [6, 7]. Here, we consider the evolution of only those solutions of Eq. (25) for which the initial disturbances in the case with $C = 0$ evolved to steadily traveling solutions of the first family of the Kuramoto–Sivashinsky equation.

As the function $\Phi(\tau)$, we chose a localized signal in the Gaussian form (Fig. 2)

$$\Phi(\tau) = \exp(-\beta(\tau - 50)^2),$$

shifted with respect to the beginning of the time axis so that the initial disturbance did not affect the further evolution of the solution. In our calculations, we assumed that $C = 1$ and $\beta = 0.1$. At such values of parameters, the values of the function $\Phi(\tau)$ outside the interval $40 < \tau < 60$ are close to zero, and Eq. (25) transforms to the Kuramoto–Sivashinsky equation.

In the calculations performed, the initial values of the real and imaginary parts of the first harmonic H_1 were assumed to be 0.5 and 0, respectively. The initial amplitudes of the remaining harmonics were assumed to be equal to zero.

Figure 3 shows the amplitudes of the first three harmonics as functions of time for the wavenumber $\alpha = 1.05$. The amplitudes of the remaining harmonics do not exceed several percent of the maximum values of the amplitude of the third harmonic. For the Kuramoto–Sivashinsky equation, the value $\alpha = 1.05$ is in the stability domain; therefore, all harmonics decay with time. Fast growth of harmonics is observed only if the function $\Phi(\tau)$ differs from zero considerably. At $\tau > 60$, the harmonics decay down to zero. Figure 4 shows the wave profiles of the surface for three typical time instants. Though the value of the function $\Phi(\tau)$ reaches the maximum at the time $\tau = 50$ (see Fig. 2), the maximum on the wave profile of the surface is reached later (cf. curves 1 and 2). As the first harmonic is substantially greater than the second one at all time instants, the shape of the wave profile is always close to a sinusoid. At $\tau > 90$, the solution almost vanishes (curve 3).

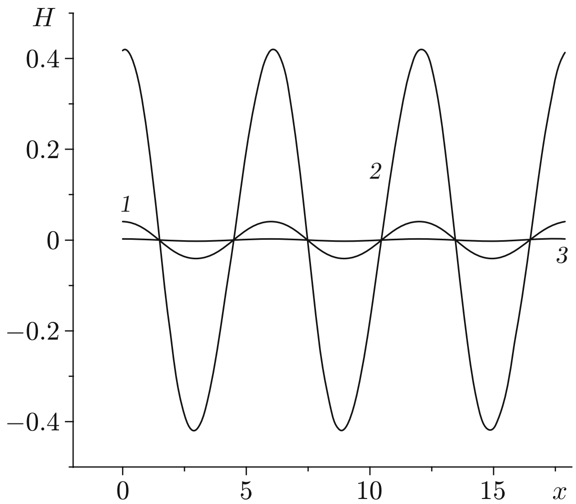

Fig. 4

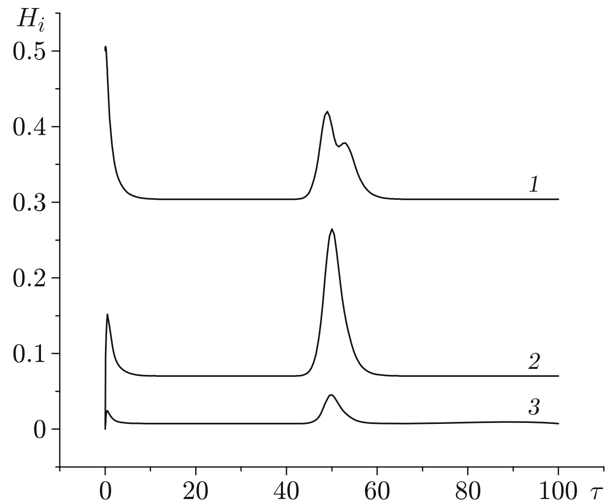

Fig. 5

Fig. 4. Wave profiles of the surface at different time instants ($\alpha = 1.05$): $\tau = 50$ (1), 54.5 (2), and 100 (3).

Fig. 5. Amplitudes of the first three harmonics versus time at $\alpha = 0.8$: H_1 (1), H_2 (2), and H_3 (3).

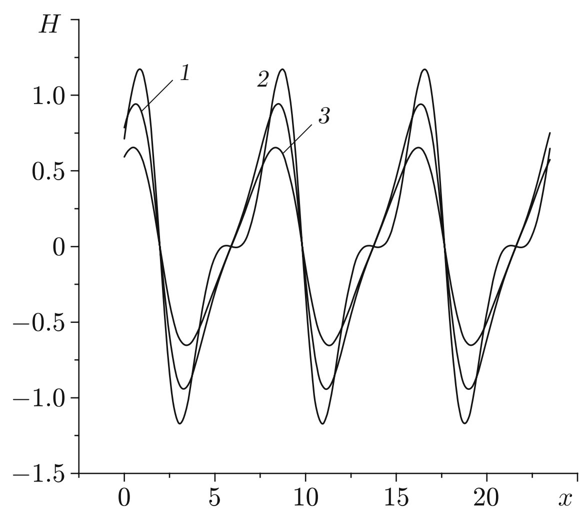

Fig. 6. Wave profiles of the surface at different time instants ($\alpha = 0.8$): $\tau = 1$ (1), 50 (2), and 100 (3).

Figure 5 shows the amplitudes of the first three harmonics versus time at $\alpha = 0.8$. It is seen that all dependences reach constant values rather rapidly at $\Phi(\tau) = 0$, i.e., a steady wave regime is formed. As the steady value of the amplitude of the first harmonic in this case is also substantially greater than the amplitude of the second harmonic, the shape of the wave profile is close to a sinusoid (curve 1 in Fig. 6). In the time interval where the function $\Phi(\tau)$ differs substantially from zero, there are peaks of all three harmonics; the values of the second and third harmonics (in percent) deviate from the steady values more significantly than the value of the first harmonic. As a result, the wave profile in the vicinity of the maximum of the function $\Phi(\tau)$ is substantially different from the sinusoid (curve 2 in Fig. 6). The presence of a local minimum and maximum of the amplitude of the first harmonic is caused by two effects: an increase in the amplitude owing to linear instability and a decrease in the amplitude owing to nonlinear actions of the remaining harmonics. All harmonics take previous constant values with time. The wave profile corresponding to these harmonics is shown in Fig. 6 (curve 3).

The calculations show that the behavior of the solutions described in the paper is typical if the wavenumbers are fairly close to the neutral wavenumber.

4. Conclusions. Equation (24) derived in the paper can be used to study the influence of an alternating electric field on weakly nonlinear disturbances in a film flow in a dielectric viscous fluid. The calculations show that wave regimes absent in the case of a freely down-flowing film can be observed even in the case of two-dimensional disturbances induced by an action limited in time.

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